Problem Set

MA18Q3-F

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Day 6

[1] Cobb–Douglas production function

Let $0 < \alpha < 1$. Consider the following production function,

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}.$$

1. Show that F has constant returns to scale.
2. Let $r + \delta = \frac{\partial F}{\partial K}$ and $w = \frac{\partial F}{\partial L}$. Compute the capital share, $(r+\delta)K/Y$ and labor share, wL/Y.
3. Define $k = K/AL$ and $y = Y/AL$. Derive the function form that relates y to k; that is, $y = f(k)$.

[2] Solow model with Cobb–Douglas production function

Consider the Solow model and assume that the production function is of Cobb–Douglas type defined above.

4. Compute the steady state capital stock, k^*, at which $\dot{k} = 0$.
5. Compute the elasticity of steady state level of k with espect to the saving rate, s:

$$\epsilon_{k^*/s} = \frac{\partial \ln k^*}{\partial \ln s},$$

which gives you a crude idea about how much percentage change of k^* is induced by one percentage increase in s. (i.e., a one percentage increase of $s \to 1.01s$ results in $k^* \to (1 + \epsilon_{k^*/s})k^*$, approximately).

6. On the balanced growth path, Y satisfies

$$Y = AL\,(k^*)^{\alpha} = A(0)L(0)e^{(g+n)t}\,(k^*)^{\alpha}.$$

Use this formula to estimate the elasticity of output, Y, with respect to the saving rate.

$$\epsilon_{Y/s} = \frac{\partial \ln Y}{\partial \ln s}.$$

Elasticity

Let $(\bar{x}, \bar{y})$ be the (perhaps, equilibrium) levels of x and y. Think of a small, positive deviation of x byΔx; x is now $\bar{x} + \Delta x$. The rate of growth is

$$\frac{\Delta x}{\bar{x}},$$

or $100 \times \left(\frac{\Delta x}{\bar{x}}\right)$ percent. (Since percent literally means $1/100$, this latter expression might be redundant.) This change in x induces a change in y:

$$\frac{\Delta y}{\bar{y}},$$

where Δy can be negative (For example, price elasticity of demand is usually negative.) Normalization gives us the definition of elasticity:

$$\frac{\frac{\Delta y}{\bar{y}}}{\frac{\Delta x}{\bar{y}}}.$$

For differentiable models, it is convenient to take the limit of $\Delta x \to 0$:

$$\frac{\frac{\Delta y}{\bar{y}}}{\frac{\Delta x}{\bar{x}}} = \frac{\bar{x}}{\bar{y}} \frac{\Delta y}{\Delta x} = \frac{\bar{x} \cdot y'(\bar{x})}{\bar{y}},$$

where y' is the derivative of y with respect to x. (Recall that the price elasticity of demand is defined by $pD'(p)/D(p)$.)

log formula

If x and y relate each other by

$$y = Cx^{\alpha},$$

the elasticity of y with respect to x is

$$\frac{xy'}{y} = \frac{x \cdot \alpha C x^{\alpha - 1}}{C x^{\alpha}} = \alpha.$$

In this case, we can obtain the elasticity α with

$$\frac{d \ln y}{d \ln x}$$

since $\ln y = \ln C + \alpha \ln x$ holds.

In general,

$$\ln y = \ln y(x) = \ln y \left(e^{\ln x}\right)$$

and thus we have

$$\frac{d \ln y}{d \ln x} = \frac{\frac{dy\left(e^{\ln x}\right)}{d(\ln x)}}{y(e^{\ln x})} = \frac{\left(\frac{d(\ln x)}{dx}\right)^{-1} \frac{dy\left(e^{\ln x}\right)}{dx}}{y(x)} = \frac{x \cdot y'(x)}{y(x)}.$$

So, $\frac{d(\ln y)}{d(\ln x)}$ is a convenient form for elasticity.[1]

[1]Although not rigorous but the "differential formula," $d(\ln y) = dy/y$ and $d(\ln x) = dx/x$, might help you memorize the elasticity formula: $\frac{d(\ln y)}{d(\ln x)} = \frac{dy/y}{dx/x} = \frac{x}{y} \cdot \frac{dy}{dx} = \frac{x \cdot y'}{y}$.